Pricing Convertible Bonds under the First-Passage Credit Risk Model

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Outline

Introduction

- A convertible bond is a corporate bond that allows the bond holder to convert the bond into the issuing firm's stock.
- Pricing convertible bonds can be intractable due to the hybrid attributes of both fixed-income securities and equities, and their complex relations to the firm's default risk.

Related Work

1. The structural credit risk model: Simulate the evolution of a firm's capital structure and specifies the conditions leading to default.[1]
    - Ingersoll (1977) and Brennan and Schwartz (1977) model the debt structure and the evolution of the firm value process for deriving partial differential equations (PDE) for pricing CBs.
    - In contrast to recent literature that models the evolution of the issuer's stock price process, these approaches model the firm value process.
        1. The firm value cannot be directly observed from the real-world markets.
        2. The jump-to-default event is hard to modeled in their approaches.

[1] Leland (1994)

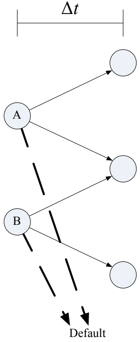
Related Work

2. The reduced-form model: Model the default probability by the credit spreads of the firm without considering the firm's capital structure.[2]
    - Hung and Wang (2002) and Chambers and Lu (2007) model the default risk by introducing the jump-to-default process modeled by the reduced-form model.
    - Their tree models make the default probabilities for the nodes at the same time step, A and B, be the same regardless of different stock prices represented by these nodes.

[2] Jarrow and Turnbull (1995)

Related Work

$$e^{-r_y \Delta t} = e^{-r\Delta t}[\,\text{Recovery value} \times P_D) + \text{Expected survival value} \times (1 - P_D\,]$$



Risk-free rate: r

Risky rate: r_y

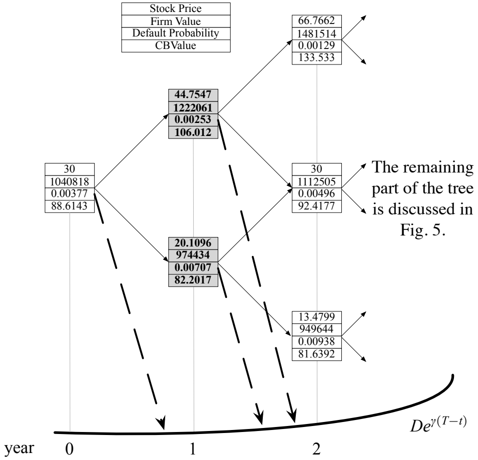
- However, a higher stock price should imply that the firm is in a better financial status and has lower default risk, and vice versa.
  - Mis-analyze the optimal strategies for exercising the conversion options and the call options embedded in CBs.
- In addition, the dilution effect is hard to describe without modeling the issuing firm's capital structure.

Main Results

- This paper proposes a tree model to analyze the relations among the stock price, the default risk, and the dilution effect via the first-passage model.[3]
  - The first-passage model models the evolution of the firm value and triggers the default event once the firm value reaches the "default boundary."
  - The equity value of the firm can be treated as a down-and-out call option on the firm value.
  - The firm value and the firm value volatility can thus be solved by calibrating the equity value and the stock price volatility by mimicking the method proposed in Merton (1974).
  - Given the firm value and the firm value volatility at each tree node, we can obtain the default probability for each node.

[3] Black and Cox (1976)

Main Results



The Lognormal Diffusion Process

- If the firm is solvent, the stock price of the issuing firm at time t, S_t, is assumed to follow the lognormal diffusion process:

$$\frac{dS(t)}{S(t)} = (r_t + \lambda'_t)\, dt + \sigma_S dZ_S,$$

where r_t denotes the risk-free short rate at time t, σ_S denotes the stock price volatility, λ'_t denotes the default intensity, and Z_S is a standard Brownian motion.

The Vasicek Short Rate Process

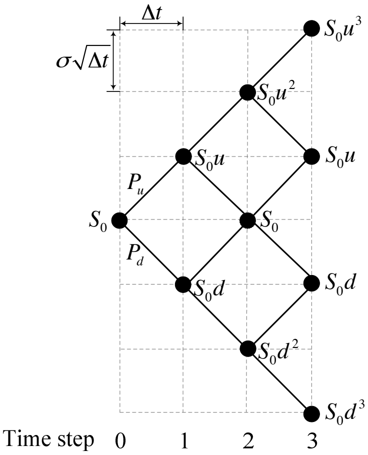
- The short rate process r_t at the two-factor model follows:

$$dr_t = a\left(b - r_t\right) dt + \sigma_r dZ_r,$$

where a denotes the mean reverting rate, b denotes the average short rate level, σ_r denotes the stock price volatility, and Z_r is a standard Brownian motion.

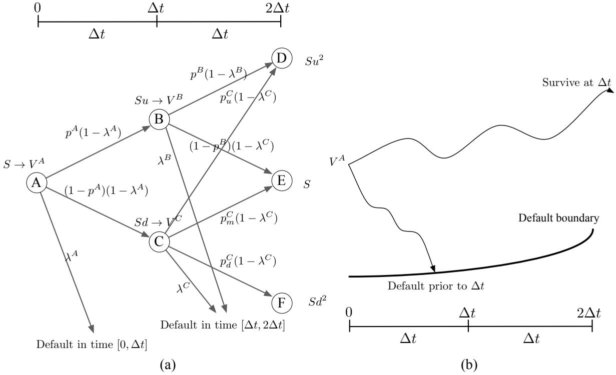
The CRR Tree

- The size of one time step is $\Delta t = T/n$.
- u, d, P_u, P_d:
  - Match the mean and variance of the stock return asymptotically.
  - $ud = 1$.
  - $P_u + P_d = 1$.



One-Factor (Stock Price) Tree Model

- Only model the stock price dynamics. r is set as a constant.
- Analyze the relationship among the stock price, the firm value, the default risk, and the optimal strategies for the embedded options.



Tree Construction

- The firm may default prior to maturity once its value hits the exogenously defined default boundary B_t.
- The equity value at time t, E_t, can be viewed as a down-and-out call option on the firm value V_t with strike price D and barrier B_t.

$$E_T = \begin{cases} (V_T - D)^+ & \text{if } V_t > B_t,\ 0 \le t \le T, \\ 0 & \text{otherwise.} \end{cases}$$

- The equity value can be evaluated by the down-and-out call option pricing formula:

$$\begin{aligned} E_t = & V_t \left[\mathrm{N}(x) - (B_t/V_t)^{[2(r-\gamma)/\sigma_v^2]+1} \mathrm{N}(y) \right] - \\ & De^{-r(T-t)} \left[\mathrm{N}(x - \sigma_v\sqrt{T-t}) - (B_t/V_t)^{[2(r-\gamma)/\sigma_v^2]-1} \mathrm{N}(y - \sigma_v\sqrt{T-t}) \right]. \end{aligned} \tag{1}$$

Tree Construction (cont.)

- The equity value E_t can be estimated by multiplying the prevailing stock price by the number of outstanding shares.
- The relation among the equity value E_t, the equity value's volatility σ_s, the firm value V_t, and firm value's volatility σ_v can be derived as follows:[4]

$$\sigma_s E_t = \frac{\partial E_t}{\partial V_t} \sigma_v V_t. \tag{2}$$

  - Thus the firm's value at time t, V_t, and its volatility σ_v can be solved by substituting E_t and σ_s into Eqs. (1) and (2).

[4] Merton (1974)

Tree Construction (cont.)

- The conditional probability λ^X for the firm to default within a time step Δt given that the stock price begins at node X can be derived as follows:

$$\begin{aligned} P(\tau \leq s|V_t) &= \mathrm{N}\left(\frac{\ln\left(B_t/V_t\right) - \left(r-\gamma-0.5\sigma_V^2\right)(s-t)}{\sigma_V\sqrt{s-t}}\right) \\ &+ \left(B_t/V_t\right)\exp\left[2\left(\frac{r-\gamma-0.5\sigma_V^2}{\sigma_V^2}\right)\right]\mathrm{N}\left(\frac{\ln\left(B_t/V_t\right) + \left(r-\gamma-0.5\sigma_v^2\right)(s-t)}{\sigma_V\sqrt{s-t}}\right). \end{aligned}$$

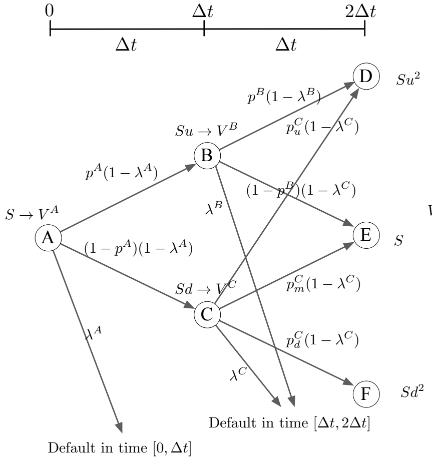
Tree Construction (cont.)

- To ensure that the expected stock price for a defaultable firm grows at the risk-free rate,[5] the branching probabilities for each node should be adjusted with the default probability of that node.
- The default intensity for an arbitrary node X, λ'^X, from the default probability λ^X:

$$e^{-\lambda'^X \Delta t} = 1 - \lambda^X \ \Rightarrow \lambda'^X = \frac{-\ln(1-\lambda^X)}{\Delta t}.$$

[5] Chambers and Lu (2007)

Tree Construction (conclude)



- Default: S jumps to 0 with probability λ^A
- Not default:
  - S moves up to Su with probability $p^A(1-\lambda^A)$
  - S moves down to Sd with probability $\left(1-p^A\right)\left(1-\lambda^A\right)$.

$$0(1-e^{-\lambda'^A\Delta t}) + Sup^A e^{-\lambda'^A\Delta t} + Sd\left(1-p^A\right)e^{-\lambda'^A\Delta t} \equiv Se^{r\Delta t}.$$

- Above, p^A can be solved to be $\frac{\exp\left(\left(r+\lambda'^A\right)\Delta t\right)-d}{u-d}$.

Backward Introduction

- The CB can be priced by the backward induction.
  - The CB value for an arbitrary node X at maturity can be expressed as
    $$\mathrm{BV}_X \equiv \max\left(\min\left(F, \mathrm{CP}_T\right), qS_T\right),$$
    where BV_X denotes the value of CBs at node X and q denotes the conversion ratio.
  - For an arbitrary node Y located at time step i prior to maturity, the CB value at node Y is
    $$\mathrm{BV}_Y \equiv \max\left(\min\left(\mathrm{CV}_Y, \mathrm{CP}_{i\Delta t}\right), qS_{i\Delta t}\right),$$

Dilution Effect

- Converting the CBs into stocks would increase the number of outstanding shares and dilute the stock value.
- The firm value V before the conversion of CBs can be expressed as:

$$V = N_B B + N_C C + N_O S^{\mathrm{BC}}.$$

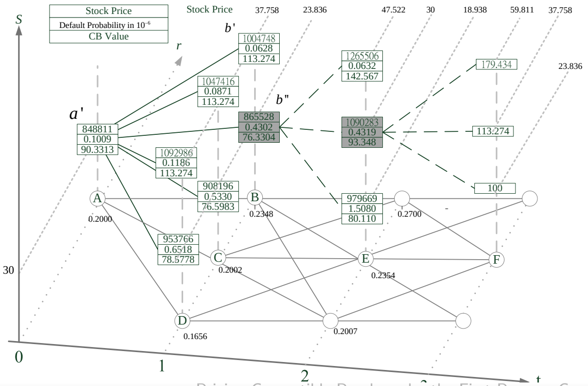
- After converting the convertible bonds into stocks, the issuer's capital structure changes and the firm value can be expressed as

$$V = N_B B + (N_O + N_C q)\, S^{\mathrm{AC}},$$

- The payoff to convert a CB is $qS^{\mathrm{AC}} = \frac{q(V - N_B B)}{N_O + N_C q}$.

Two-Factor (Stock Price & Short Rate) Tree

- Let the short rate r_t follow the Vasicek model.
- 3D tree for the stock price and the short rate.
  - The underlying tree models the short rate process. Tuning branching probabilities to match correlations.



Tuning branching probabilities to match correlations.

	r_t Moves Upward	Middle	Downward
S_t Moves Upward	$P_u p + \epsilon$	$P_m p$	$P_d p - \epsilon$
Moves Downward	$P_u(1-p) - \epsilon$	$P_m(1-p)$	$P_d(1-p) + \epsilon$

An Empirical Case

- Combine the one-factor tree model with the Hull-White interest rate model to construct a two-factor tree to price CB subject to the interest rate.[6]
- A six-year zero-coupon CB issued by Lucent
  1. $S_0 = 15.006$, $\sigma_S = 0.353836$, $T = 6$, $F = 100$, $q = 5.07524$, and $\rho = -0.1$.
  2. The CB cannot be called for the first three years, and the call prices are 94.205, 96.098, and 98.030 for the fourth, the fifth, and the sixth year, respectively.
  3. The risk-free zero coupon rates are 5.969%, 6.209%, 6.373%, 6.455%, 6.504%, and 6.554% for the first, the second, ..., and the sixth year.

[6] The details for the tree construction is available in the paper.

An Empirical Case (conclude)

- A six-year zero-coupon CB issued by Lucent[7]
  4. From the financial report of Lucent: the numbers of outstanding stocks and convertible bonds are 642,062,656 and 2,290,000, respectively.
  5. The payment of straight bond due at maturity is estimated by the value of liability minus the face value of convertible bonds; which is 20,195,000,000.
- Results of Hung and Wang (2002): 90.4633 and that of Chambers and Lu (2007): 90.83511
- Our pricing results are 90.1903 (without considering the dilution effect) and 89.223 (considering the dilution effect), which is much closer to the market price 88.706 than the above two pricing results.

[7] Hung and Wang (2002) and Chambers and Lu (2007)

Conclusions

- This paper develops a CB pricing method based on the structural credit risk model.
- By taking advantages provided by the structural credit risk model, three features can be dealt with in our tree model:
  1. The default probabilities for nodes with different stock prices (implying different financial status of the firm) will be different.
  2. The dilution effect can be described.
  3. The recovery rate can be endogenous defined. (In this talk, we omit this part for simplicity.)
- The preliminary results show that the price of our tree model is much closer to the market price than those of the previous researches.