A Multi-Phase, Flexible, and Accurate Lattice for Pricing Complex Derivatives with Multiple Market Variables

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Outline

Introduction

- Sophisticated derivatives are constantly being structured to fit the needs of markets.
    - Addressing their sophisticated features significantly increases the difficulty of pricing them.
- Besides, the importance of some factors, like sovereign risk or credit risk, which are overlooked in primitive derivatives pricing models, is being recognized as key due to recent financial crises.
    - E.g., Vulnerable options.[1]

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[1] Klein (1996); Klein and Inglis (2001).

Need for Numerical Methods

- Most complex derivatives have no analytical formulas for their prices, particularly when there is more than one market variable.
- As a result, these derivatives must be priced by numerical methods such as lattice.
- However, the nonlinearity error of lattices due to the nonlinearity of the derivative's value function could lead to oscillating prices.[2]

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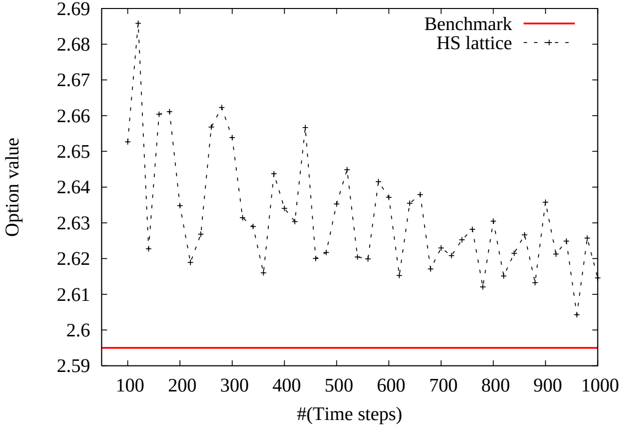
[2] Figlewski and Gao (1999).

Oscillation Problem



Multivariate Derivatives

- Multivariate derivatives elevate the pricing difficulty to a new level compared with that of univariate derivatives.
- The correlations between the market variables must be carefully handled.
- Otherwise, invalid branching probabilities may result.[3]

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[3] Zvan et al. (2003).

Related Works

- Rubinstein (1994) builds a three-dimensional lattice called pyramid for two correlated assets.
    - But his lattice is not flexible enough to suppress the nonlinearity error.
- Hull and White (1994) build a lattice by assuming that the processes of market variables are independent first, and then adjusting the branching probabilities to reflect the correlations.
    - But the branching probabilities can be negative.
- Andricopoulos et al. (2007) propose a quadrature method to handle multiple assets.
    - This method can suppress the nonlinearity error.
    - But it is not as efficient as the lattice in handling continuous sampling features, like the American exercise feature and the continuous barrier options.[4]

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[4] Dai and Lyuu (2010).

Main Results

- This paper proposes a multi-phase methodology to build multivariate lattices for pricing complex derivatives with small nonlinearity errors.
- We adopt Hull and White (1990b)'s orthogonalization method to handle the correlations between market variables.
  - The orthogonalization transforms the original, correlated processes into uncorrelated ones.
- The multi-phase method builds the lattice for the transformed, uncorrelated processes.

The Lognormal Diffusion Process

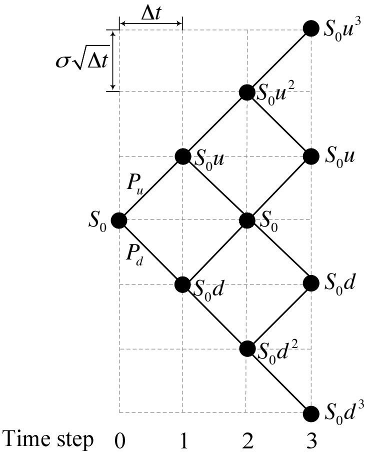
- The market variable follows a lognormal diffusion process with a constant volatility σ and a constant riskless rate r:

  $$\frac{dS(t)}{S(t)} = r dt + \sigma dz(t),$$

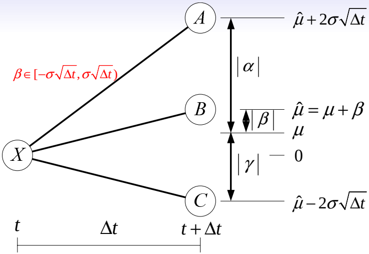
  where $dz(t)$ denotes a standard Brownian motion.

The CRR Lattice

- The size of one time step is $\Delta t = T/n$.
- u, d, P_u, P_d:
  - Match the mean and variance of the stock return asymptotically.
  - $ud = 1$.
  - $P_u + P_d = 1$.



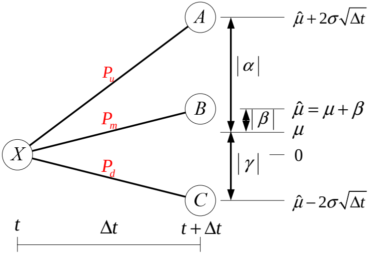
Trinomial Structure



The branching probabilities for the node X

$$
\begin{aligned}
&\beta \equiv \hat{\mu} - \mu,\\
&\alpha \equiv \hat{\mu} + 2\sigma\sqrt{\Delta t} - \mu = \beta + 2\sigma\sqrt{\Delta t},\\
&\gamma \equiv \hat{\mu} - 2\sigma\sqrt{\Delta t} - \mu = \beta - 2\sigma\sqrt{\Delta t},\\
&\hat{\mu} \equiv \ln\left(s(B)/s(X)\right).
\end{aligned}
$$

Trinomial Structure (concluded)



The branching probabilities for the node X

$$P_u\alpha + P_m\beta + P_d\gamma = 0,$$
$$P_u(\alpha)^2 + P_m(\beta)^2 + P_d(\gamma)^2 = \text{Var},$$
$$P_u + P_m + P_d = 1.$$

Price Oscillation Problem

- Price oscillation problem is mainly due to the nonlinearity error.
    - Introduced by the nonlinearity of the option value function.
- The solution to the nonlinearity error:
    - Make a price level of the lattice coincide with the location where the option value function is highly nonlinear.

Lattice Construction

- We demonstrate our multi-phase method for two correlated market variables.
    - For more than two market variables, follow the same procedure.
- To simultaneously handle the correlations and make our lattice match the critical locations, Hull and White (1990b)'s orthogonalization method is revised as follows:
    - Order the market variables first.
    - Then orthogonalize the original, correlated processes into uncorrelated ones.
- Then build the multivariate lattice for the uncorrelated processes and make this lattice match the critical locations.

Orthogonalization

- Before the transformation, the market variables are so ordered that the *i*-th coordinates of the critical locations depend only on the first *i* of the market variables.
- E.g., The two correlated market variables are ordered so that S_1 is followed by S_2 when
  - The first coordinates of the critical locations are functions of S_1.
  - The second coordinates are functions of S_1 and S_2.

Orthogonalization (cont.)

- Let S_1 and S_2 be represented as follows:

$$\begin{aligned} dS_1 &= \mu_1 dt + \sigma_1 dz_1, \\ dS_2 &= \mu_2 dt + \sigma_2 dz_2. \end{aligned}$$

  - The correlation between dz_1 and dz_2 is ρ.

Orthogonalization (cont.)

- It is a standard fact that dz_2 can be decomposed into a linear combination of dz_1 and another independent Brownian motion dz:

$$dz_2 = \rho\, dz_1 + \sqrt{1-\rho^2}\, dz.$$

- The differential forms of S_1 and S_2 can be written as

$$\begin{bmatrix} dS_1 \\ dS_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_1 & 0 \\ \sigma_2 \rho & \sigma_2 \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz \end{bmatrix}.$$

- Now, transform S_1 and S_2 into two uncorrelated processes X_1 and X_2:

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} \mu_1/\sigma_1 \\ \frac{-\rho\,\mu_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{\mu_2}{\sigma_2\sqrt{1-\rho^2}} \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dz_1 \\ dz \end{bmatrix}. \quad (1)$$

Orthogonalization (concluded)

- Integrate both sides of Eq. (1) to yield

$$\begin{aligned} X_1(t) &= \frac{S_1(t) - S_1(0)}{\sigma_1}, \\ X_2(t) &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{S_2(t) - S_2(0)}{\sigma_2} - \rho X_1(t) \right), \end{aligned}$$

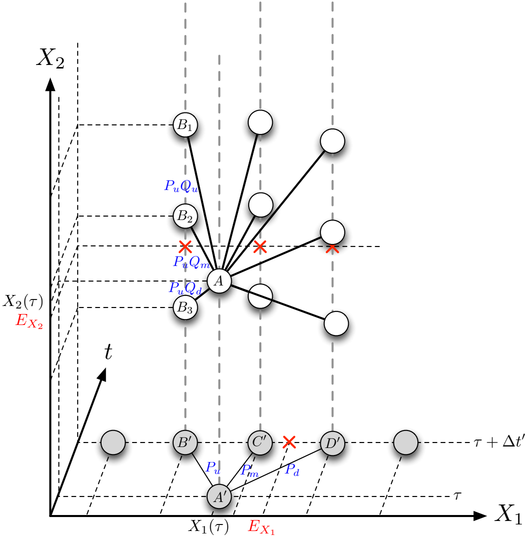
  where $X_1(0) = X_2(0) = 0$ for convenience.

- $S_1(t)$ and $S_2(t)$ can be backed out of $X_1(t)$ and $X_2(t)$ thus:

$$\begin{aligned} S_1(t) &= S_1(0) + \sigma_1 X_1(t), \\ S_2(t) &= S_2(0) + \sigma_2 \left(\sqrt{1-\rho^2} X_2(t) + \rho X_1(t) \right). \end{aligned}$$

Core Ideas of Our Multi-Phase Branch Construction

- Consider a bivariate lattice that approximates the evolution of two uncorrelated processes $X_1(t)$ and $X_2(t)$.
- The construction contains two phases.
    1. Lattice for $X_1(t)$.
    2. Lattice for $X_2(t)$.



A Bivariate Lattice: Two Correlated Market Variables

- We now built a bivariate lattice to price vulnerable barrier options with the strike price K and the barrier $B(t) = Be^{-\gamma(T-t)}$.
  - The two market variables: the stock price, $S(t)$, and the firm's asset value, $V(t)$.
- The default boundary for the firm's asset value at time t, $D^*(S(t), t) = De^{-r(T-t)} + c(S(t), t)$.[5]
- In this setup, the option holder receives

$$c(S(t), t)/D^*(S(t), t)$$

of the firm's asset value when the firm defaults.

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[5] Klein and Inglis (2001).

A Bivariate Lattice: Two Correlated Market Variables (cont.)

- $S(t)$ and $V(t)$ are both assumed to follow the lognormal diffusion processes.
- We first order the two processes: $\ln S(t)$ is the first process and $\ln V(t)$ the second.
- We then apply the orthogonalization process to obtain two uncorrelated processes.

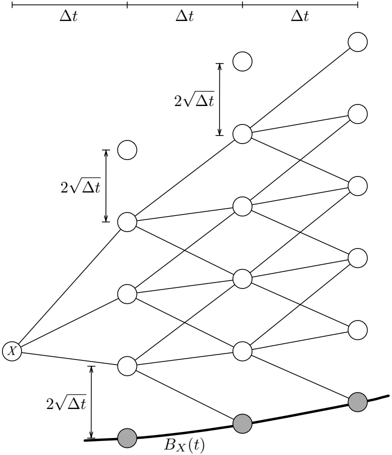
$$
\begin{aligned}
dX(t) &= \frac{1}{\sigma_S}\left(r - \frac{\sigma_S^2}{2}\right) dt + dz_S, \\
dY(t) &= \frac{1}{\sqrt{1-\rho^2}}\left(-\frac{\rho}{\sigma_S}\left(r - \frac{\sigma_S^2}{2}\right) + \frac{1}{\sigma_V}\left(r - \frac{\sigma_V^2}{2}\right)\right) dt + dz.
\end{aligned}
$$

A Bivariate Lattice: Two Correlated Market Variables (cont.)

- The $X(t)$-lattice is first built.
- $B(t)$ will be transformed to $B_X(t)$ on the $X(t)$-lattice via:

$$B_X(t) = \frac{1}{\sigma_S} \left(\ln B(t) - \ln S(0) \right).$$

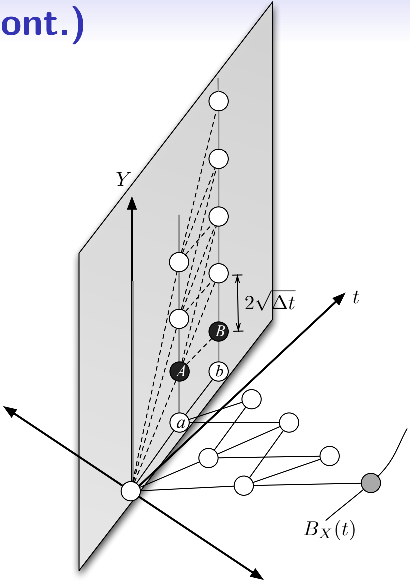
- The lattice starts by placing gray nodes on the barrier to reduce the nonlinearity error.
- All the other nodes are then laid from the gray nodes upward and downward.



A Bivariate Lattice: Two Correlated Market Variables (cont.)

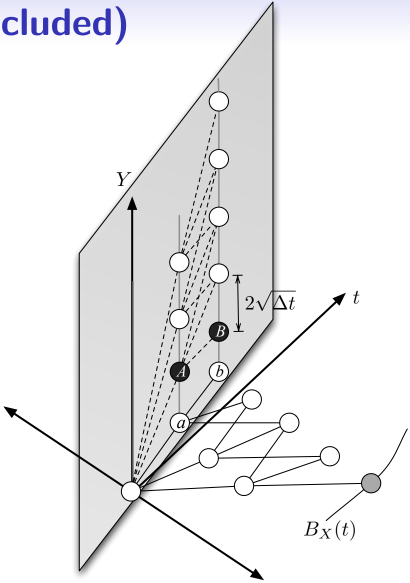
- The second phase builds the $Y(t)$-lattice first.
- $D^*(t)$ will be transformed to $D^*_Y(t)$ on the $Y(t)$-lattice via:

$$D^*_Y(t) = \left(\frac{\ln D^*(t) - \ln V(0) - \rho X(t)}{\sigma_V \sqrt{1-\rho^2}} \right)$$

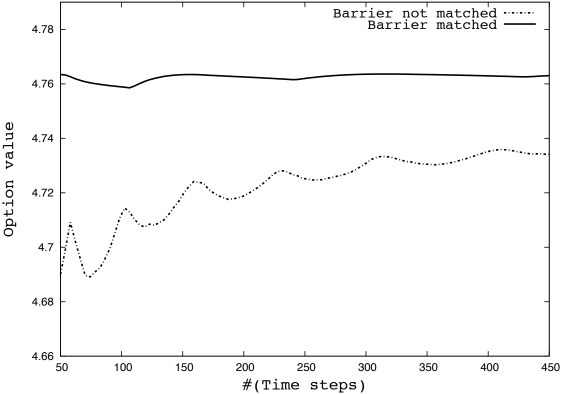


A Bivariate Lattice: Two Correlated Market Variables (concluded)

- Once we have $D_Y^*(t)$, the lattice starts by placing nodes on this default boundary (the black nodes).
- In the end of the second phase, the $Y(t)$-lattice is added "on top of" the $X(t)$-lattice to form the bivariate lattice.



Convergence of the Vulnerable Barrier Call Option



Evaluating Defaultable Bonds under Short Rate Models

	Defaultable Zero Bonds with an Exogenous Default Boundary			
Face value (F)	`Lattice`	`Formula`[6]	Relative errors	Default-free bonds
2000	1924.8	1924.8	−0.002%	1925.0
2500	2404.0	2404.4	−0.015%	2406.2
3000	2874.9	2876.2	−0.045%	2887.4

Call/ Put prices	Callable bonds		Putable bonds		Straight bonds	
	Defaultable	Default-free	Defaultable	Default-free	Defaultable	Default-free
3030	3016.6	3029.4	3063.6	3063.8	3017.6	3034.8
3035	3017.2	3033.0	3068.4	3068.7	3017.6	3034.8
3040	3017.5	3034.4	3073.2	3073.5	3017.6	3034.8
3050	3017.6	3034.8	3082.9	3083.2	3017.6	3034.8

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[6] Briys and De Varenne (1997).

Conclusions

- This paper proposes a flexible multi-phase method to build a multivariate lattice for pricing derivatives accurately.
- To simultaneously handle the correlations and make the lattice match the critical locations:
  1. The market variables are first properly ordered.
  2. Then the original, correlated market variables are transformed into uncorrelated ones by orthogonalization.
- A multivariate lattice is then built for the transformed, uncorrelated processes.
- Numerical results show that our methodology can be applied to price a wide range of complex financial contracts efficiently and accurately.